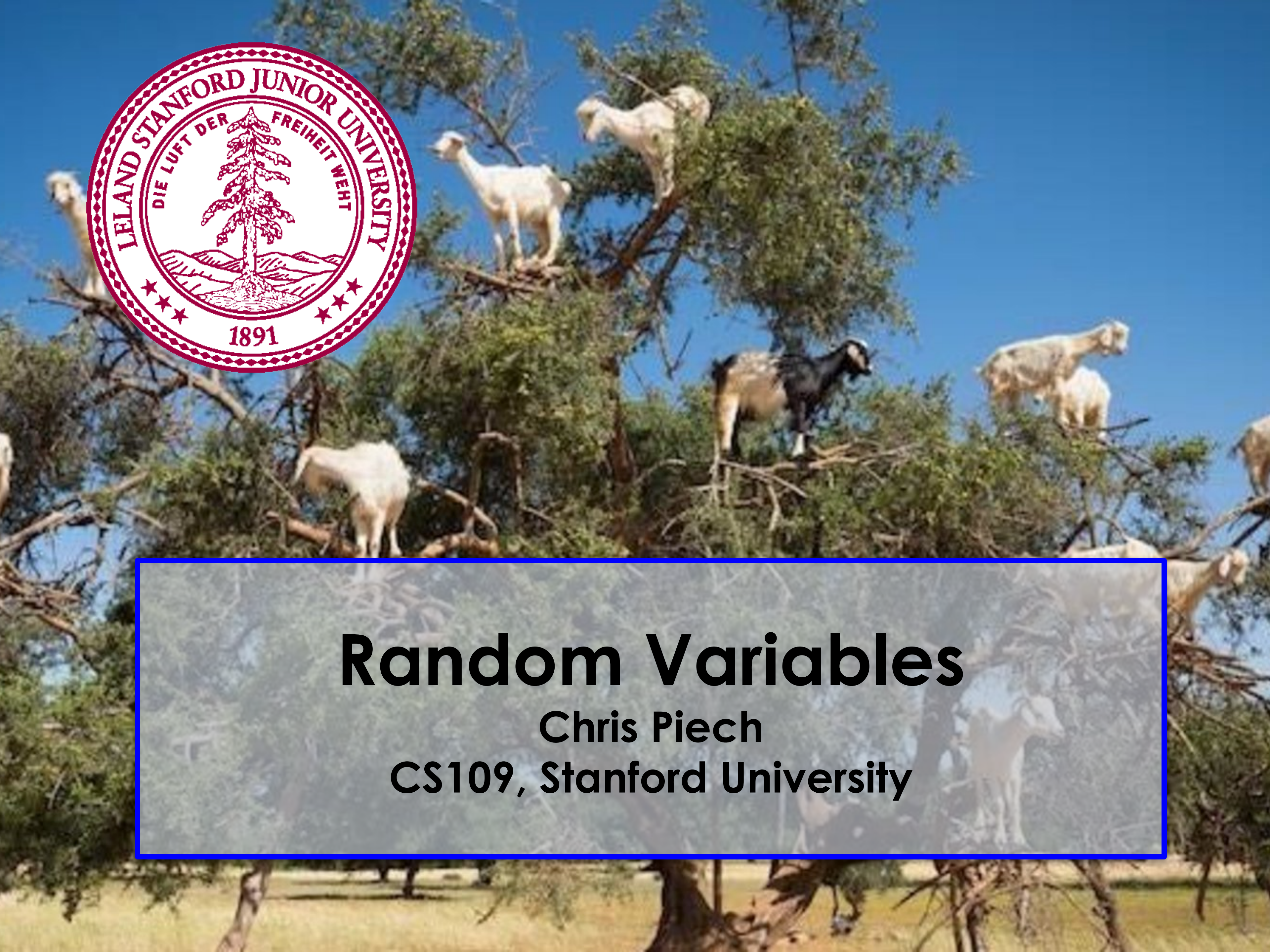


Random Variables

Chris Piech
CS109, Stanford University

Let's Play a Game

- Game set-up
  - We have a fair coin (come up "heads" with p = 0.5)
  - Let *n* = number of coin flips ("heads") before first "tails"
  - You win $\$2^n$
- How much would you pay to play?

Learning Goals

1. Be able to use conditional independence
2. Be able to define a random variable (R.V.)
3. Be able to use + produce a PMF of a R.V.
4. Be able to calculate the expectation of the R.V.

Conditional Paradigm

- Recall:

$$P(A\ B) = P(B\ A\ )$$

$$P(A\ B) = P(A \mid B\ )\ P(B)$$

Conditional Paradigm

- For any events A, B, and E, you can condition consistently on E, and all formulas still hold:

$$P(A\,B \mid E) = P(B\,A \mid E)$$

$$P(A\,B \mid E) = P(A \mid B\,E)\,P(B \mid E)$$

- Can think of E as "everything you already know"
- Formally, $P(\bullet \mid E)$ satisfies 3 axioms of probability

BAE's Theorem?

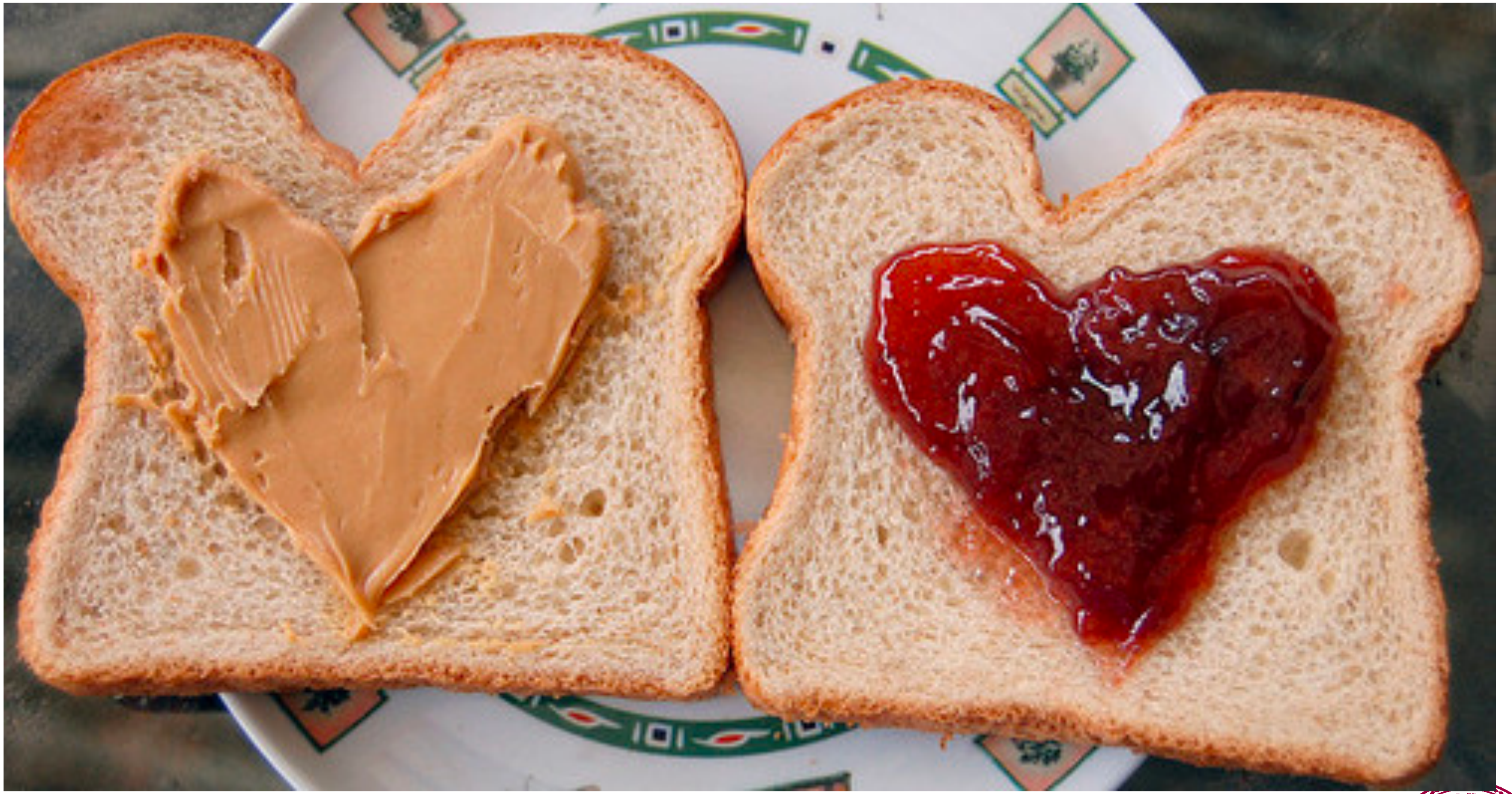
$$P(A \mid B\ E) = \frac{P(B \mid A\ E)\ P(A \mid E)}{P(B \mid E)}$$

In the conditional paradigm, the formulas of probability are preserved.

Two Great Tastes

Conditional Probability

Independence



Conditional Independence

- Two events E and F are called **conditionally independent given G**, if

$$P(EF|G) = P(E|G)P(F|G)$$

- Or, equivalently if:

$$P(E|FG) = P(E|G)$$

Independence relationships can change with conditioning.

If E and F are independent, that does not mean they will still be independent given another event G.

There is additional reading about this in the course reader. You will explore this more in depth in CS228

NETFLIX

And Learn

Netflix and Learn

What is the probability that a user will watch Life is Beautiful?

$P(E)$

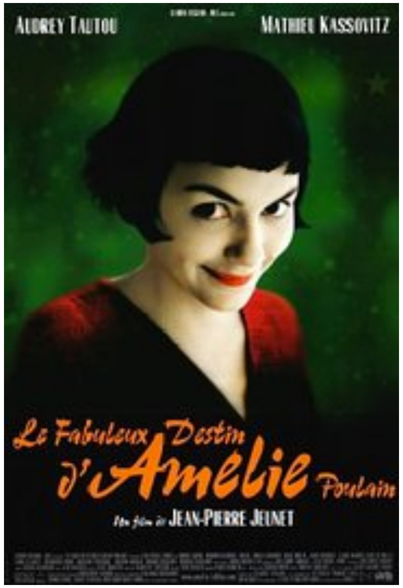
$$P(E) = \lim_{n \to \infty} \frac{n(E)}{n} \approx \frac{\#\text{people who watched movie}}{\#\text{people on Netflix}}$$

$P(E)$ = 10,234,231 / 50,923,123 = 0.20

Netflix and Learn

What is the probability that a user will watch Life is Beautiful, given they watched Amelie?

$P(E|F)$



$$P(E|F) = \frac{P(EF)}{P(F)} = \frac{\#\text{people who watched both}}{\#\text{people who watched } F}$$

$P(E|F) = 0.42$

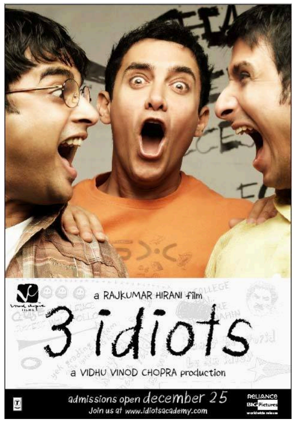
Conditioned on liking a set of movies?

Netflix and Learn

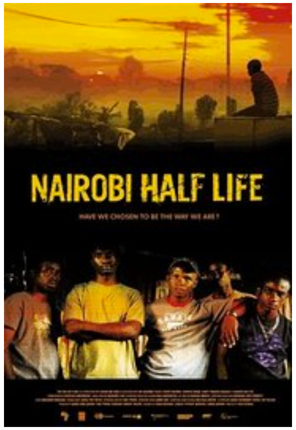
Each event corresponds to liking a particular movie

E_1

E_2



E_3



E_4

$$P(E_4 | E_1, E_2, E_3)?$$

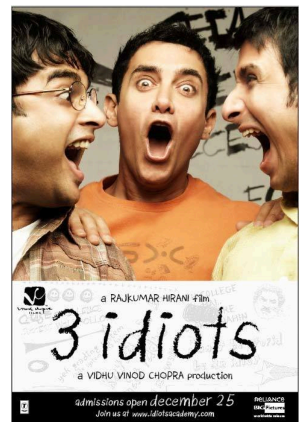
Is E_4 independent of E_1,E_2,E_3?

Netflix and Learn

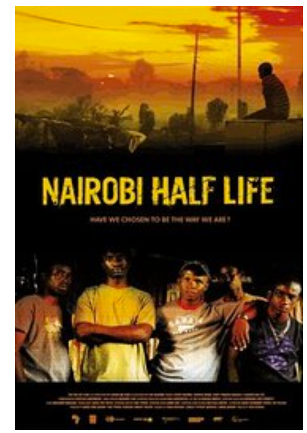
Is E_4 independent of E_1,E_2,E_3?

E_1

E_2



E_3



E_4

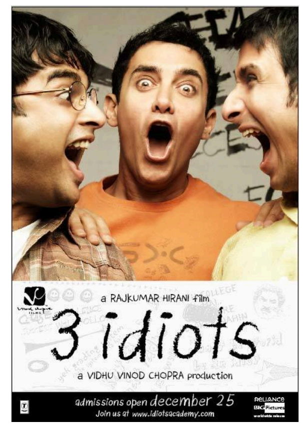
$$P(E_4|E_1, E_2, E_3) \stackrel{?}{=} P(E_4)$$

Netflix and Learn

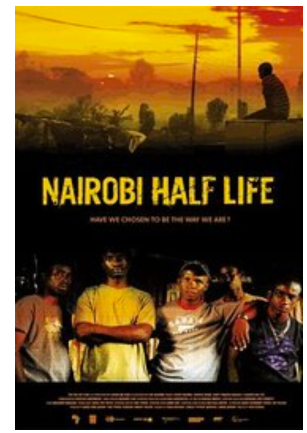
Is E_4 independent of E_1, E_2, E_3?

E_1

E_2



E_3



E_4

$$P(E_4 | E_1, E_2, E_3) = \frac{P(E_1 E_2 E_3 E_4)}{P(E_1 E_2 E_3)}$$

Netflix and Learn

- What is the probability that a user watched four particular movies?
  - There are 13,000 titles on Netflix
  - The user watches 30 random titles
  - E = movies watched include the given four.
- Solution:

Watch those four

Choose 24 movies not in the set

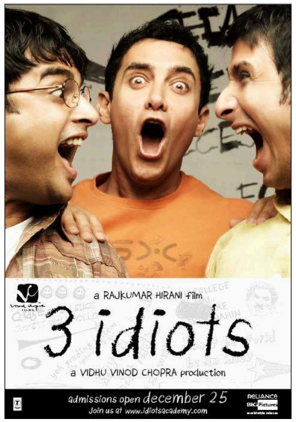
$$P(E) = \frac{\binom{4}{4}\binom{12996}{24}}{\binom{13000}{30}} = 10^{-11}$$

Choose 30 movies from netflix

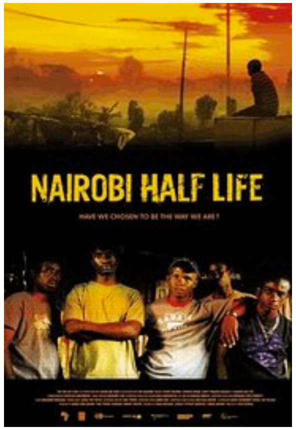
Netflix and Learn

E_1

E_2



E_3



E_4

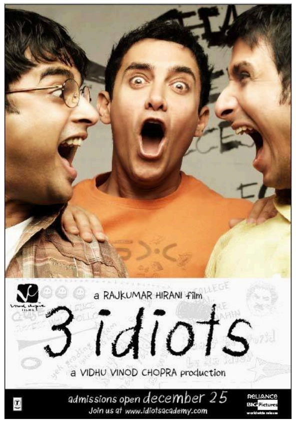
Netflix and Learn

K_1

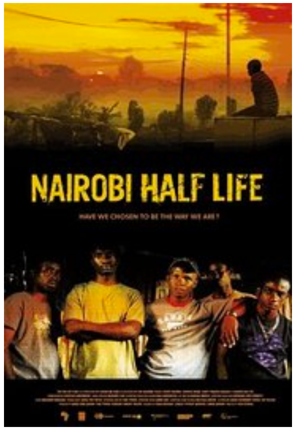
Like foreign emotional comedies

E_1

E_2



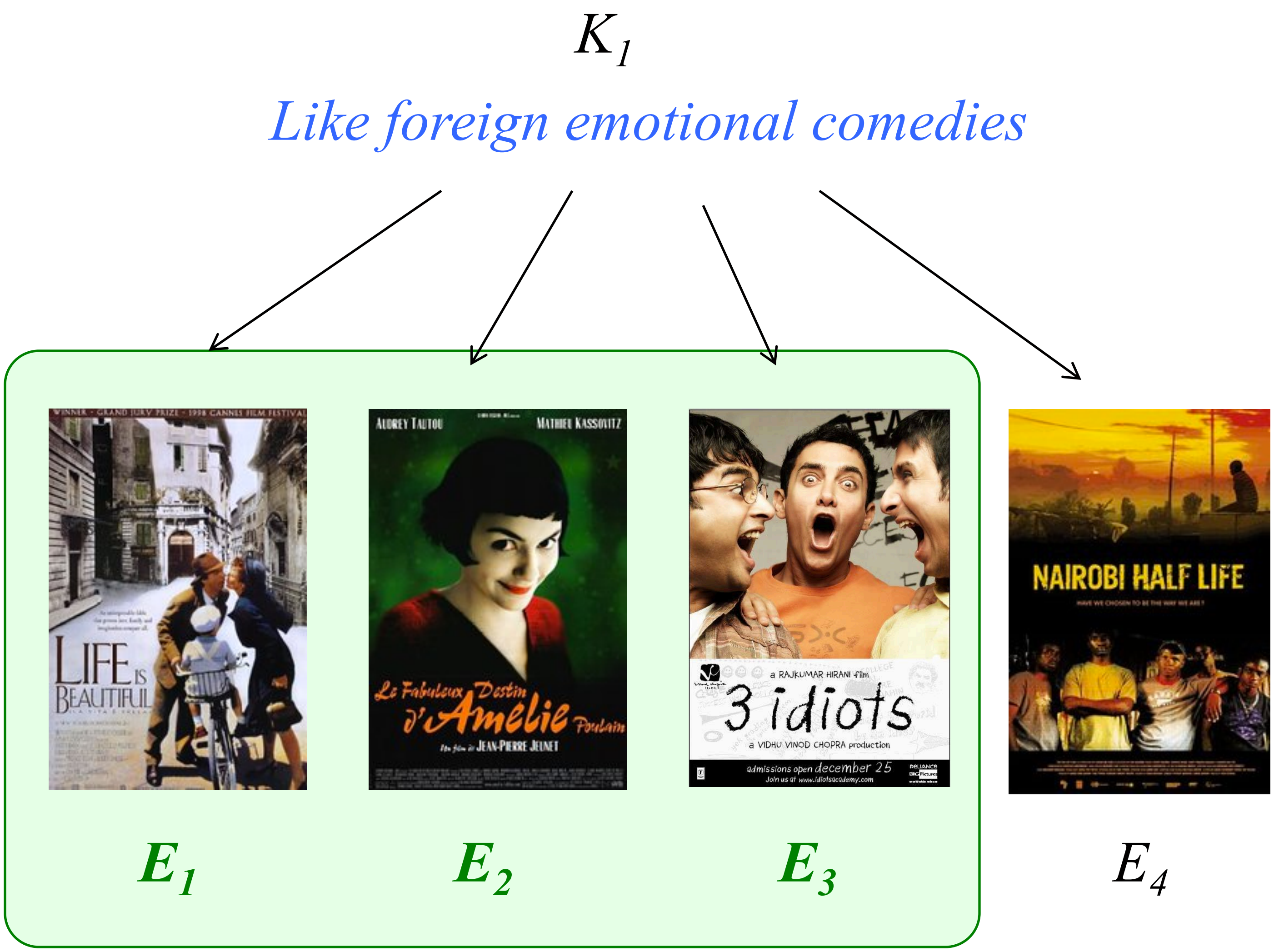
E_3



E_4

Assume E_1, E_2, E_3 and E_4 are conditionally independent given K_1

Netflix and Learn



Assume E_1, E_2, E_3 and E_4 are conditionally independent given K_1

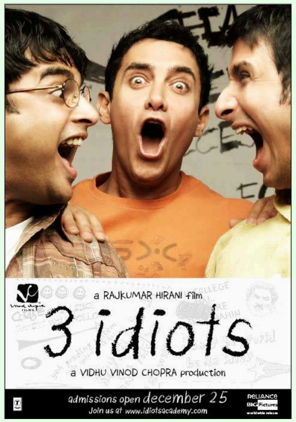
Netflix and Learn

K_1

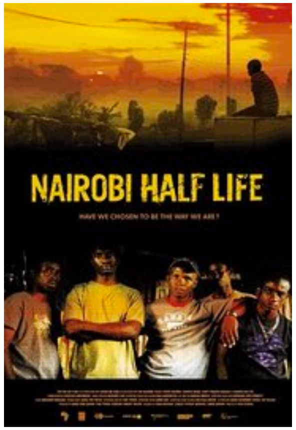
Like foreign emotional comedies

E_1

E_2



E_3



E_4

Assume E_1, E_2, E_3 and E_4 are conditionally independent given K_1

Netflix and Learn

K_1

Like foreign emotional comedies

$$P(E_4 | E_1, E_2, E_3, K_1) = P(E_4 | K_1)$$

E_1 E_2 E_3 E_4

Assume E_1, E_2, E_3 and E_4 are conditionally independent given K_1

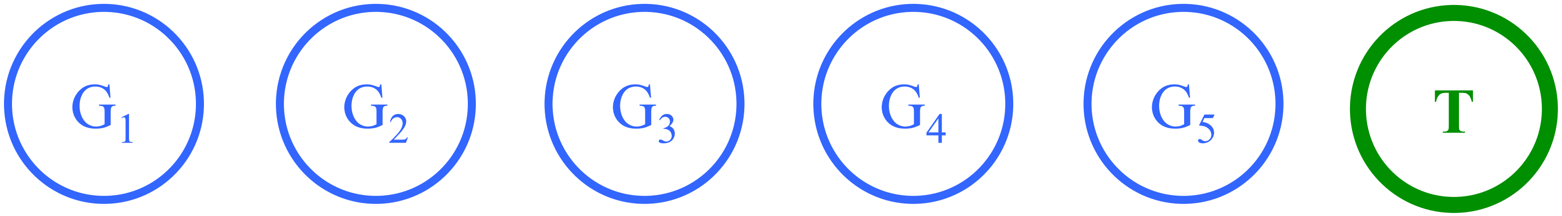
Conditional independence is a practical, real world way of decomposing hard probability questions.

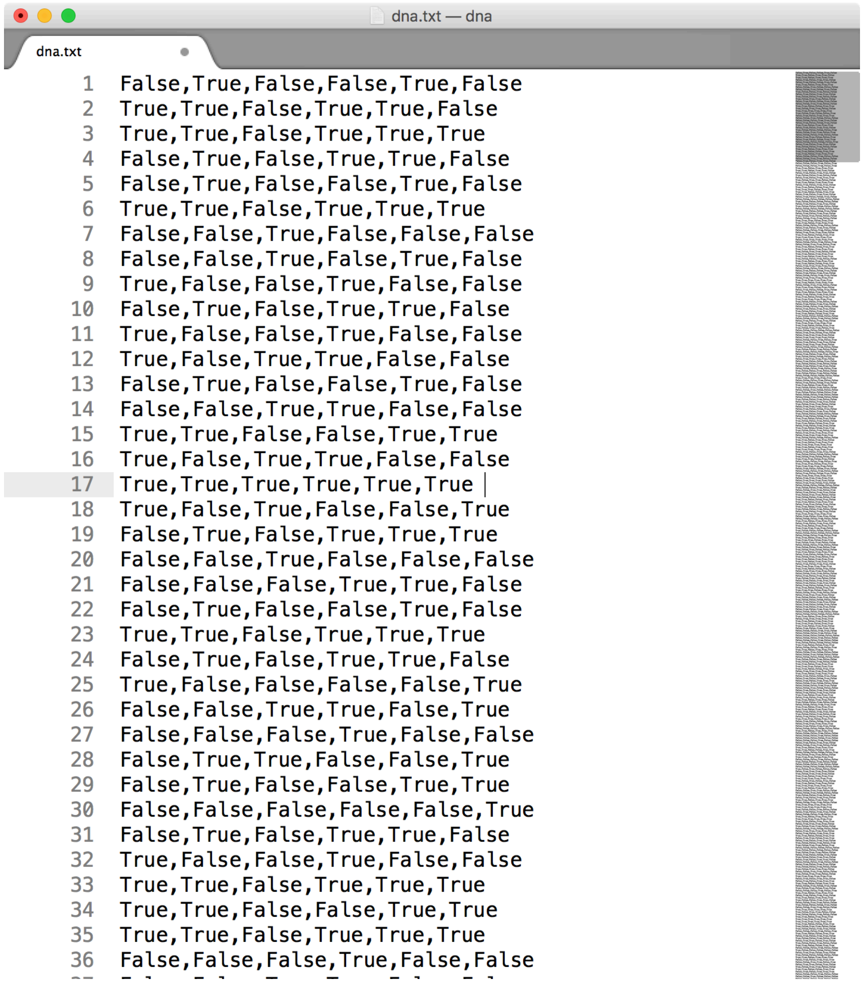
Big Deal

"Exploiting conditional independence to generate fast probabilistic computations is one of the main contributions CS has made to probability theory"

-Judea Pearl wins 2011 Turing Award, *"For fundamental contributions to artificial intelligence through the development of a calculus for probabilistic and causal reasoning"*


G_1
G_2
G_3
G_4
G_5
T






```
False,True,False,False,True,False
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True,True,False,True,True,True
False,False,True,False,False,False
False,False,True,False,True,False
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True,False,False,True,False,False
True,True,False,True,True,True
True,True,False,False,True,True
True,True,False,True,True,True
False,False,False,True,False,False
```

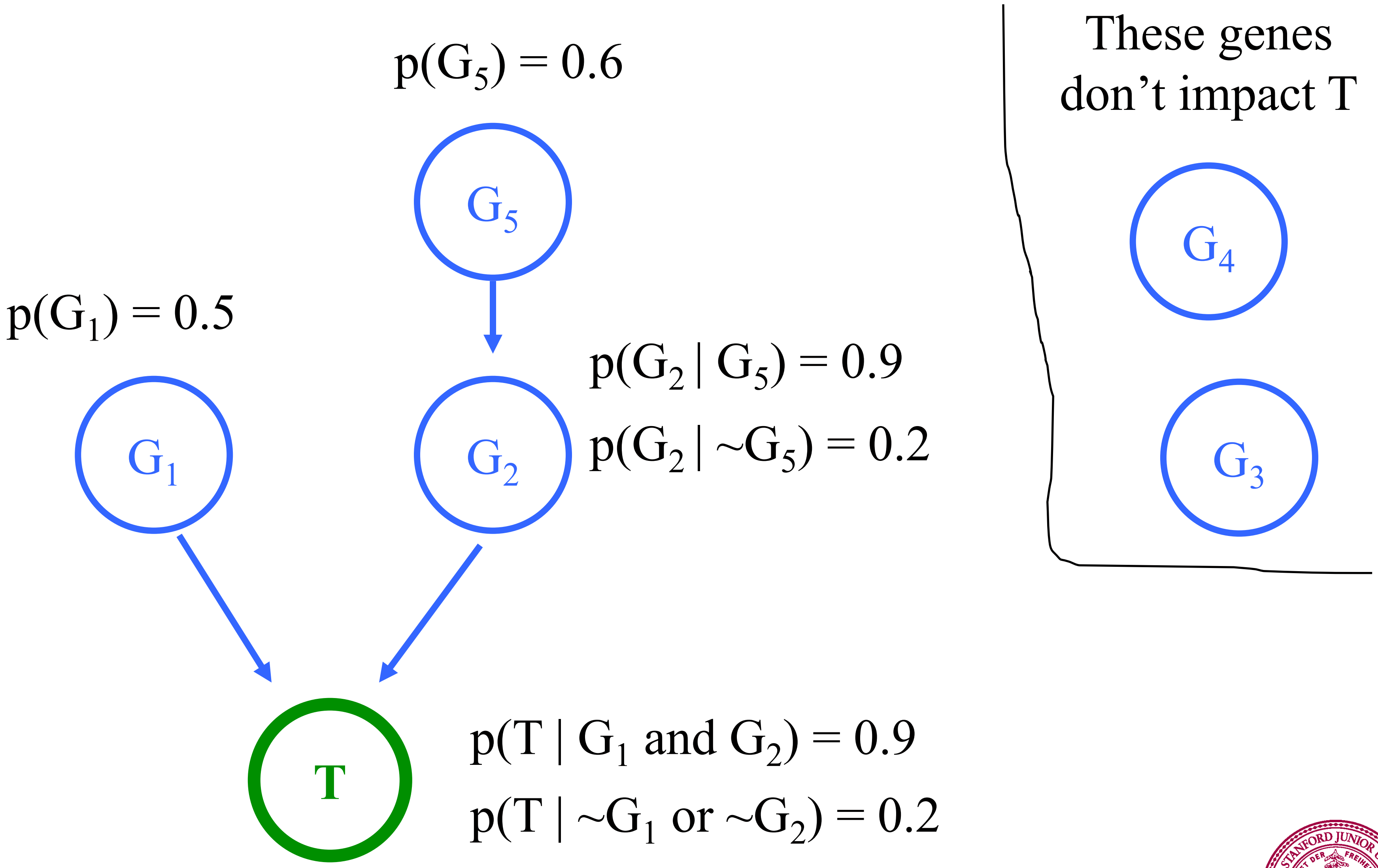
100,000 samples

6 observations per sample

Discovered Pattern

```
Piech-2:dna piech$ python findStructure.py
size data =  100000
p(G1) = 0.500
p(G2) = 0.545
p(G3) = 0.299
p(G4) = 0.701
p(G5) = 0.600
p(T) = 0.390
p(T and G1) = 0.291 , P(T)p(G1) = 0.195
p(T and G2) = 0.300 , P(T)p(G2) = 0.213
p(T and G3) = 0.116 , P(T)p(G3) = 0.117
p(T and G4) = 0.273 , P(T)p(G4) = 0.273
p(T and G5) = 0.309 , P(T)p(G5) = 0.234
T is independent of G3
T is independent of G4
G1 is independent of G2
G1 is independent of G5
T is independent of G5 | G2
```

Use Independence to Hypothesize



Next: Ubiquitous formalism

Remember Learning to Code?

name

value

type

```
int a = 5;
double b = 4.2;
bit c = 1;
choice d = medium;
```

$$z \in \{\text{high}, \text{medium}, \text{low}\}$$

Random Variable

- A **Random Variable** is a real-valued function defined on a sample space
- Example:
  - 3 fair coins are flipped.
  - Y = number of “heads” on 3 coins
  - Y is a random variable
  - $P(Y = 0) = 1/8$ (T, T, T)
  - $P(Y = 1) = 3/8$ (H, T, T), (T, H, T), (T, T, H)
  - $P(Y = 2) = 3/8$ (H, H, T), (H, T, H), (T, H, H)
  - $P(Y = 3) = 1/8$ (H, H, H)
  - $P(Y \geq 4) = 0$

Pirates of the Random Variables

```
int a = 5;
```

A is the number of pirate ships in our future armada.

$$A \in \{1, 2, \ldots, 10\}$$

```
double b = 4.2;
```

B is the amount of money we get after we defeat Blackbeard.

$$B \in \mathbb{R}^+$$

```
bit c = 1;
```

C is 1 if we successfully raid Isla de Muerta. 0 otherwise.

$$C \in \{0, 1\}$$

It is confusing that both random variables and events use the same notation

Random variables and events are two *different* things

We can define an event to be a particular assignment to a random variables

Example Random Variable

- A coin flip has 2 possible outcome. Consider *n* coin flips, each which independently come up heads with probability *p*
  - Recall:

$$P(2 \text{ heads}) = \binom{n}{2} p^2 (1-p)^{n-2}$$

$$P(3 \text{ heads}) = \binom{n}{3} p^3 (1-p)^{n-3}$$

  - Y = number of "heads" on *n* flips

$$Y \in \{1, 2, \ldots, n\}$$

$$P(Y = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

* Pro tip: no coin works like this… but many real world binary events do

Simple Game

- Urn has 11 balls (3 blue, 3 red, 5 black)
  - 3 balls drawn. +$1 for blue, -$1 for red, $0 for black
  - Y = total winnings
  - P(Y = 0) = $\left[\binom{5}{3} + \binom{3}{1}\binom{3}{1}\binom{5}{1}\right] \Big/ \binom{11}{3} = \frac{55}{165}$
  - P(Y = 1) = $\left[\binom{3}{1}\binom{5}{2} + \binom{3}{2}\binom{3}{1}\right] \Big/ \binom{11}{3} = \frac{39}{165}$ = P(Y = -1)
  - P(Y = 2) = $\binom{3}{2}\binom{5}{1} \Big/ \binom{11}{3} = \frac{15}{165}$ = P(Y = -2)
  - P(Y = 3) = $\binom{3}{3} \Big/ \binom{11}{3} = \frac{1}{165}$ = P(Y = -3)

Fun with Random Variables

- Probability Mass Function:

$$P(X = a)$$

- Expectation:

$$E[X]$$

- Variance:

$$\text{Var}(X)$$

Learning goals for today

1. Probability Mass Function

All the different assignments to a random variable make a function

If this is a number

$$P(Y = 2)$$

Then this is a number

For example Y is the number of heads in 5 coin flips

If this is a variable

$$P(Y = k)$$

Then this is a function

For example Y is the number of heads in 5 coin flips

Random Variables -> Functions

$$P(Y = k)$$

$$k = 5$$

$$0.03125$$

For example Y is the number of heads in 5 coin flips

Random Variables -> Functions

$$P(Y = k)$$

```
private double eventProbability(int k) {
        int ways = choose(N, k);
        double a = Math.pow(P, k);
        double b = Math.pow(P, N-k);
        return ways * a * b;
}

private static final int N = 5;
private static final double P = 0.6;
```

For example Y is the number of heads in 5 coin flips

If a random variable is discrete we call this function the Probability Mass Function

Probability Mass Function

Let X be a random variable that represents the result of a **single dice roll**. X can take on the values {1, 2, 3, 4, 5, 6}

$$P(X = x)$$

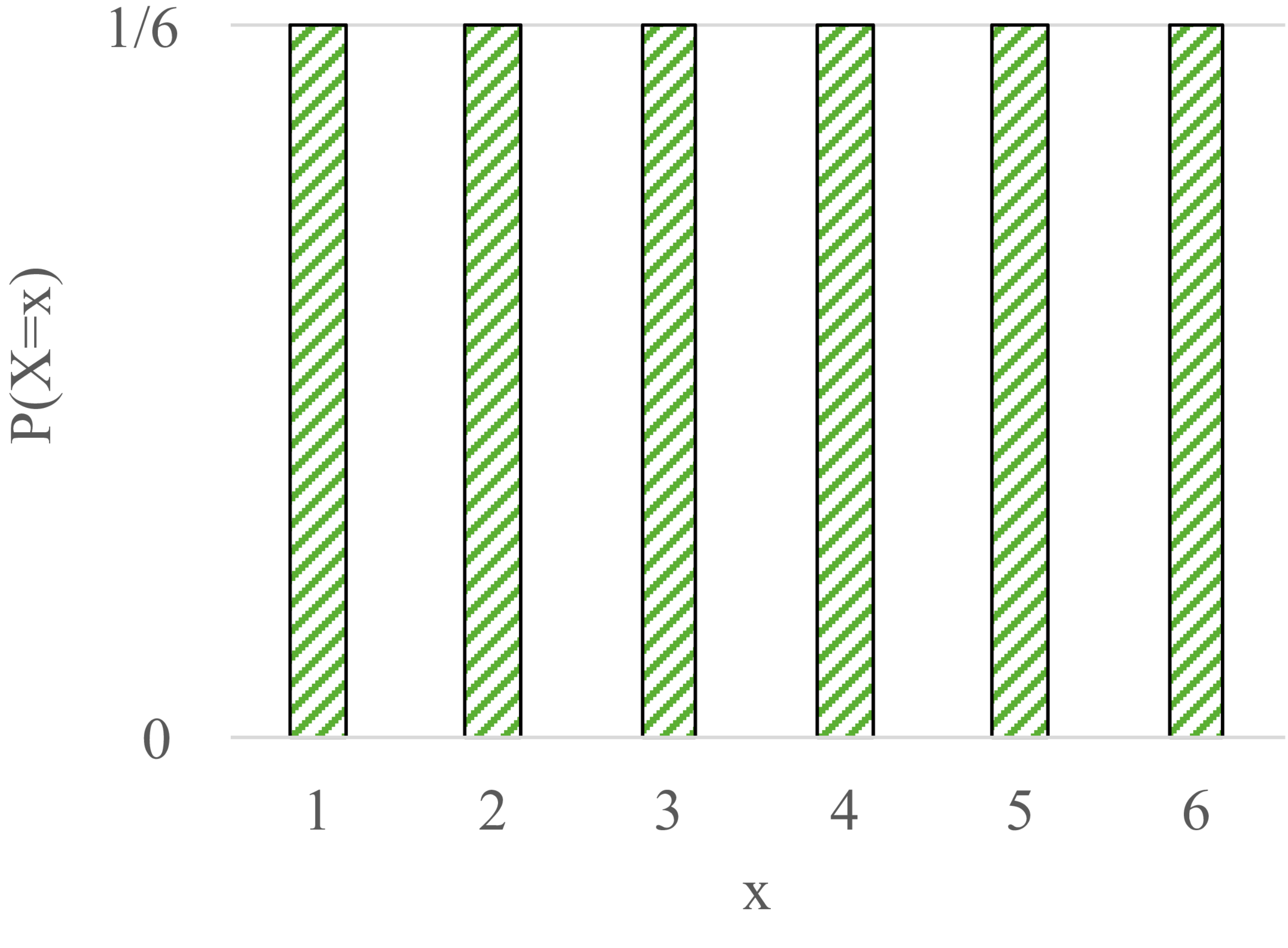
$$p(x)$$

This is shorthand notation for the PMF

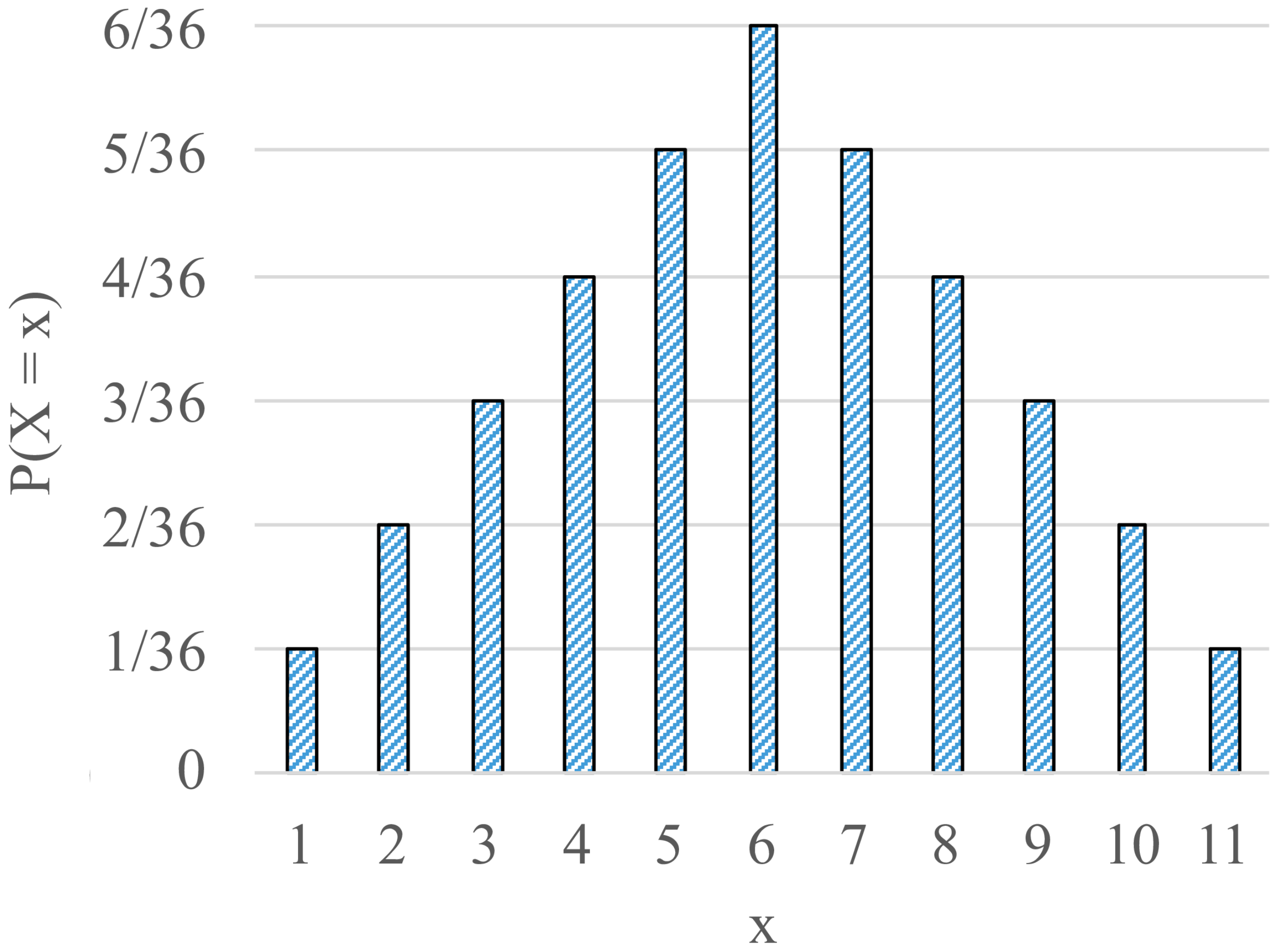
$$p_X(x)$$

This is also shorthand notation for the PMF

PMF For a Single 6 Sided Dice



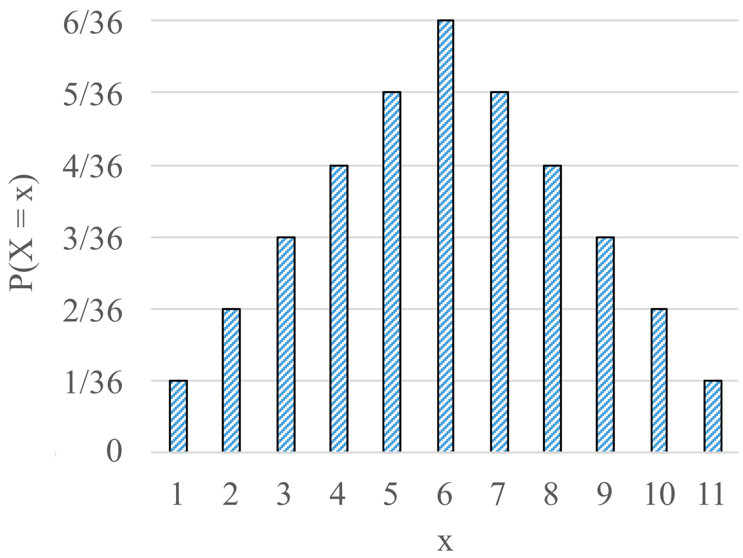
PMF for the sum of two dice



PMF as an Equation

$$P(X = x) = \begin{cases} \frac{x}{36} & \text{if } x \in \mathbb{R} \text{ , } 0 \leq x \leq 6 \\ \frac{12-x}{36} & \text{if } x \in \mathbb{R} \text{ , } x \leq 7 \\ 0 & \text{else} \end{cases}$$

Again, this is the probability for the sum of two dice



2. Expectation

Expected Value

- The Expected Values for a discrete random variable X is defined as:

$$E[X] = \sum_{x:p(x)>0} x \cdot p(x)$$

- Note: sum over all values of x that have $p(x) > 0$.

- Expected value also called: *Mean*, *Expectation*, *Weighted Average*, *Center of Mass*, *1st Moment*

Expected Value

- Roll a 6-Sided Die. X is outcome of roll
  - p(1) = p(2) = p(3) = p(4) = p(5) = p(6) = 1/6
- E[X] = $1\left(\frac{1}{6}\right)+2\left(\frac{1}{6}\right)+3\left(\frac{1}{6}\right)+4\left(\frac{1}{6}\right)+5\left(\frac{1}{6}\right)+6\left(\frac{1}{6}\right)=\frac{7}{2}$
- Y is random variable
  - P(Y = 1) = 1/3, P(Y = 2) = 1/6, P(Y = 3) = 1/2
- E[Y] = 1 (1/3) + 2 (1/6) + 3 (1/2) = 13/6

Lying with Statistics

"There are three kinds of lies:
lies, damned lies, and statistics"

– *Mark Twain*

- School has 3 classes with 5, 10 and 150 students
- Randomly choose a class with equal probability
- X = size of chosen class
- What is E[X]?
  - E[X] = 5 (1/3) + 10 (1/3) + 150 (1/3)
    = 165/3 = 55

Lying with Statistics

"There are three kinds of lies:
lies, damned lies, and statistics"

– *Mark Twain*

- School has 3 classes with 5, 10 and 150 students
- Randomly choose a student with equal probability
- Y = size of class that student is in
- What is E[Y]?
  - $E[Y] = 5\,(5/165) + 10\,(10/165) + 150\,(150/165)$
    $= 22635/165 \approx 137$
- Note: E[Y] is students' perception of class size
  - But E[X] is what is usually reported by schools!

More on Expectation

Properties of Expectation

- **Linearity**:

$$E[aX + b] = aE[X] + b$$

  - Consider X = 6-sided die roll, Y = 2X – 1.
  - E[X] = 3.5 E[Y] = 6

- **Expectation of a sum** is the sum of expectations

$$E[X + Y] = E[X] + E[Y]$$

- **Unconscious statistician**:

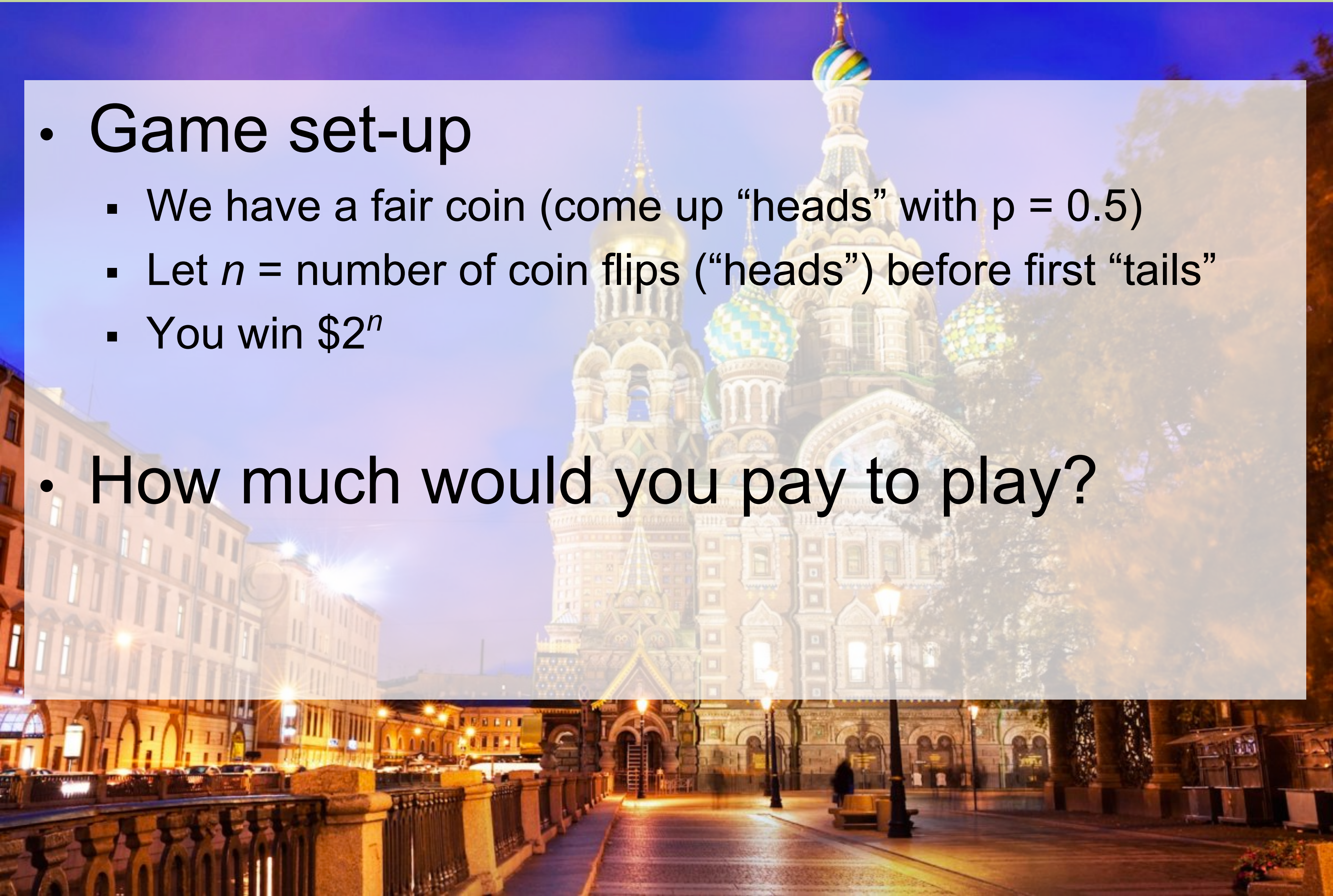
$$E[g(x)] = \sum_x g(x)p(x)$$

Wonderful

St Petersburg

- Game set-up
  - We have a fair coin (come up “heads” with p = 0.5)
  - Let n = number of coin flips (“heads”) before first “tails”
  - You win \$$2^n$

- How much would you pay to play?



St Petersburg

- Game set-up
  - We have a fair coin (come up "heads" with p = 0.5)
  - Let *n* = number of coin flips ("heads") before first "tails"
  - You win $\$2^n$
- How much would you pay to play?
- Solution
  - Let X = your winnings
  - E[X] = $\left(\frac{1}{2}\right)^1 2^0 + \left(\frac{1}{2}\right)^2 2^1 + \left(\frac{1}{2}\right)^3 2^2 + \left(\frac{1}{2}\right)^4 2^3 + \ldots \quad = \sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^{i+1} 2^i$

    $$= \sum_{i=0}^{\infty} \frac{1}{2} = \infty$$
  - I'll let you play for $1 thousand... but just once! Takers?

St Petersburg + Reality

- What if Chris has only \$65,536?
  - Same game
  - If you win over \$65,536 I leave the country.
- Solution
  - Let X = your winnings
  - E[X] $= \left(\frac{1}{2}\right)^1 2^0 + \left(\frac{1}{2}\right)^2 2^1 + \left(\frac{1}{2}\right)^3 2^2 + \left(\frac{1}{2}\right)^4 2^3 + \dots$

$$= \sum_{i=0}^{k} \left(\frac{1}{2}\right)^{i+1} 2^i \text{ s.t. } k = \log_2(65,536)$$

$$= \sum_{i=0}^{16} \frac{1}{2} = 8.5$$